Rank-based M-Estimation for Tail Dependence and Independence

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Objective

Using an iid sample $(X_1, Y_1), ..., (X_n, Y_n) \sim (X, Y)$, estimate the joint tail probability

$$\mathbb{P}(X \geq q_1, Y \geq q_2)$$

where

- X, Y are possibly dependent
- q_1, q_2 are large thresholds

Univariate Extreme Value Theory (see [1])

- With an iid sample $X_1, ..., X_n \sim F$, we cannot use sample quantiles to estimate $\bar{F}(q) := 1 - F(q)$ if $q > X_{(n)}$ (the estimate would be 0)
- Instead, assume F belongs to a *max-domain of attraction* (most common continuous distributions do)
- Then as $u \to F^{-1}(1)$,

$$\frac{\bar{F}(u+x)}{\bar{F}(u)} \longrightarrow (1 + \gamma x/\sigma)^{-1/\gamma}, \quad \sigma > 0, \gamma \in \mathbb{R}$$

- Choose u high enough so that
  1. u is close enough to the tail of F
  2. There is enough data above u to estimate the parameters σ, γ
- Usually, u is the $(1 - k/n)$–th sample quantile, where $k = k_n$ is an intermediate sequence ($k \to \infty$, $k/n \to 0$)
- Then use estimator

$$\bar{F}(u+x) \approx \hat{\bar{F}}(u)(1 + \hat{\gamma}x/\hat{\sigma})^{-1/\hat{\gamma}},$$

where $\hat{\sigma}$ and $\hat{\gamma}$ are, e.g., MLE's

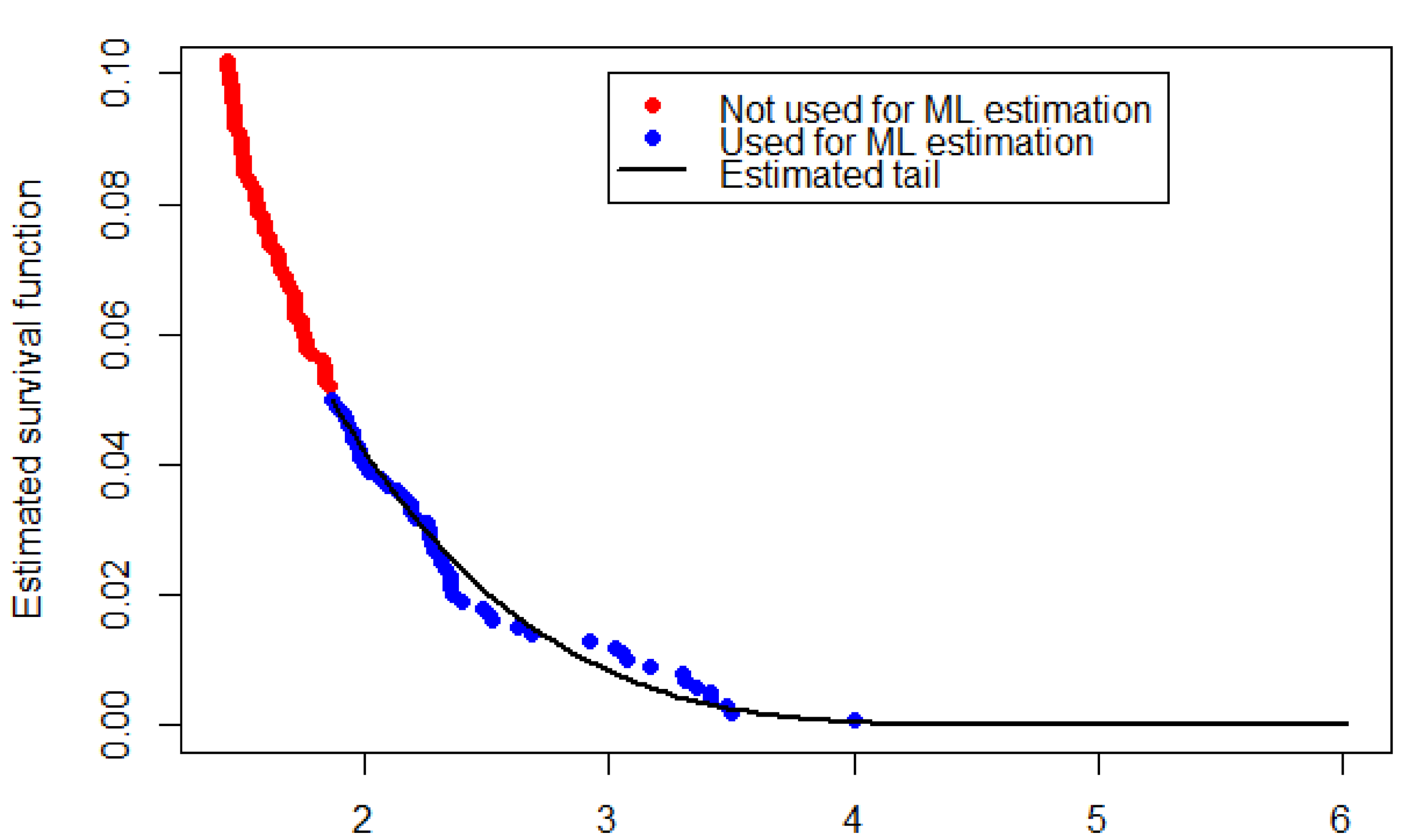

Figure 1: Estimation of the tail of a sample of size 1000 from the t_{10} distribution, using MLE's of σ and γ and the 95–th sample percentile as threshold u.

General Strategy

- Multivariate EVT is much harder since there is no unique parametric model for the *joint* tail
- **Can estimate the probability that X and Y simultaneously exceed their respective high (unknown) quantiles...**
- ... and use univariate methods to estimate these quantiles

Our Assumption

- Uniformly over (x, y) such that $x^2 + y^2 = 1$,

$$\frac{\mathbb{P}(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty)}{q(t)} = c(x, y) + O(t^\alpha), \quad t \downarrow 0,$$

  for some $\alpha > 0$
- This is a classic assumption in multivariate EVT ([2, 3])
- The *survival tail copula function* c describes the shape of the joint tail, whereas q characterizes the strength of tail dependence
- Compared to existing methods, we make no smoothness assumptions on c (e.g. differentiability, cf [2])
- Simple cases:
  - Perfect independence: $q(t) = t^2$, $c(x, y) = xy$
  - Perfect (positive) dependence: $q(t) = t$, $c(x, y) = x \wedge y$
- If $q(t) = o(t)$, X and Y are said to be *asymptotically independent*, otherwise *asymptotically dependent*

Nonparametric Method

- Choose intermediate sequence k and "estimate" c by

$$\hat{c}_n(x, y) = \frac{n^{-1}\sum_{i=1}^n \mathbb{1}\{R_X(X_i) \geq n + 1 - kx, R_Y(Y_i) \geq n + 1 - ky\}}{q(k/n)}$$

Theorem 1

If $nq(k/n) \to \infty$ and $nq(k/n)(k/n)^{2\alpha} \to 0$, $\sqrt{nq(k/n)}(\hat{c}_n - c)$ converges weakly to a Gaussian process W

- in $\ell^\infty_{\text{Loc}}([0, \infty)^2)$, under asymptotic independence.
- in the topology of hypi convergence for locally bounded functions, under asymptotic dependence (see [4]).

Parametric Method

- Why? In general, the estimated function $\hat{c}_n$ is not an admissible survival tail copula function
- Assume we have a family $\{c_\theta\}, \theta \in \Theta \subseteq \mathbb{R}^p$, of admissible functions that contains the true function c_{θ_0}
- Would like to estimate θ_0 by minimizing

$$\left\| \int gc_\theta - \int g\hat{c}_n \right\|$$

  for some integrable, $\mathbb{R}^q$–valued ($q > p$) weight function g
- Instead, minimize

$$\left\| \sigma \int gc_\theta - \int g \times nq(k/n)\hat{c}_n \right\|,$$

  over all $(\theta, \sigma) \in \Theta \times (0, \infty)$, and get estimator $\hat{\sigma}_n$ of $nq(k/n)$ for free!

Theorem 2

1. Under assumptions of Theorem 1 and smoothness conditions on the map $\theta \mapsto \int gc_\theta$,

$$\sqrt{nq(k/n)}\left(\left(\hat{\theta}_n, \frac{\hat{\sigma}_n}{nq(k/n)}\right) - (\theta_0, 1)\right)$$

   is asymptotically normal.
2. For $(x, y) \in (0, \infty)^2$, under further smoothness conditions on the map $\theta \mapsto c_\theta(x, y)$,

$$\sqrt{nq(k/n)}\left(\frac{\hat{\sigma}_n c_{\hat{\theta}_n}(x, y)/n}{\mathbb{P}(F_1(X) \geq 1 - kx/n, F_2(Y) \geq 1 - ky/n)} - 1\right)$$

   is asymptotically normal.

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