Dependence Estimation in Spatial Extremes

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Extreme value theory

- Extrapolate outside the range of data and estimate properties of the underlying distribution *beyond the observed regions*
- Mathematically, EVT estimates the tail of the d.f. F of a random variable
- Univariate methods fairly well developed
- Multivariate and spatial methods: dependence modeling is challenging
- Paper: Bivariate $\Rightarrow$ Spatial

Motivating data

- $n \approx 18\,000$ rainfall measurements at $d = 40$ locations in southern Australia (Le et al., 2018)
- Typical problem: estimate

$$\mathbb{P}(\text{rainfall at location } j > x_j, \text{for each } j \in J),$$

for large values x_j and finite collection of locations J

- Both the locations J and the levels x_j could be unobserved
- Marginal tails are estimated by univariate EVT methods, so we focus on the dependence structure

Spatial dependence model

- Inverted Brown–Resnick process on $\mathbb{R}^2$ (or a subset)
- Constructed from normalized maxima of iid Gaussian processes over a "shrinking" region (Brown and Resnick, 1977)
- Parametrized by a *variogram function* $\gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$
- If the marginal distributions are correctly transformed, variogram of $Z := \{Z(u) : u \in \mathbb{R}^2\}$ is

$$\gamma(u, u') = \mathbb{V}\mathrm{ar}(Z(u) - Z(u'))$$

- Most popular model: *fractal variogram*

$$\gamma_\vartheta(u, u') = (\|u - u'\|/\beta)^\alpha,$$

$\vartheta := (\alpha, \beta) \in (0, 2) \times (0, \infty)$

Identification via bivariate tails

- For an IBR process Z, and any pair of locations $s = (u, u') \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$t^{-2\theta^{(s)}} \mathbb{P}(F_u(Z(u)) \geq 1 - tx, F_{u'}(Z(u')) \geq 1 - ty) \longrightarrow (xy)^{\theta^{(s)}},$$

  as $t \downarrow 0$
- $\theta^{(s)} := \Phi(\sqrt{\gamma(u, u')}/2) \in (1/2, 1]$
- So $\{$all the $\theta^{(s)}\} \Rightarrow \gamma$
- But knowing $\gamma \in \{\gamma_\vartheta\}$, $\{$a small number of $\theta^{(s)}\} \Rightarrow \vartheta$
- So only need to estimate the pairwise parameters $\theta^{(s)}$
- Note: every pair is *asymptotically independent*

Bivariate approach

- Fix an observed pair $s = (u, u')$, say $(X, Y) = (Z(u), Z(u'))$
- Wish to estimate

$$c(x, y) := \lim_{t \downarrow 0} q(t)^{-1} \mathbb{P}(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty)$$

- The function c characterizes bivariate tail dependence much more general than the bivariate margins of IBR processes (asympt. dep. or indep.)

Estimation of c

- (iid) observations $(X_1, Y_1), \ldots, (X_n, Y_n)$
- Recall

$$c(x, y) := \lim_{t \to 0} q(t)^{-1} \mathbb{P}(F_1(X) \geq 1 - tx, F_2(Y) \geq 1 - ty)$$

- For a sequence $t_n \downarrow 0$,

$$\widehat{c}_n(x, y) := q(t_n)^{-1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1} \left\{ \widehat{F}_1(X_i) \geq 1 - t_n x, \widehat{F}_2(Y_i) \geq 1 - t_n y \right\},$$

  where $\widehat{F}_j$ are the empirical d.f.

- Issue: q is unknown

Estimation of c

- We know $c(x,y) = c_\theta(x,y) := (xy)^\theta$, for $\theta \in (1/2, 1]$
- Estimate θ by

$$\min_{\theta,\sigma} \left\| \sigma \int_{[0,T]^2} g(x,y) c_\theta(x,y) dxdy - q(t_n) \int_{[0,T]^2} g(x,y) \widehat{c}_n(x,y) dxdy \right\|,$$

  for any weight function $g : [0, T]^2 \to \mathbb{R}^p$

- We use $g = (\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_p})$
- Minimzer $(\widehat{\theta}, \widehat{\sigma})$
- If $\widehat{c}_n \approx c$, then hopefully $\widehat{\theta} \approx \theta$ and $\widehat{\sigma} \approx q(t_n)$

Spatial estimation

- Get estimates $\widehat{\theta}^{(s)}$ for each observed pair s (or a subset thereof)
- Recall that if $s = (u, u')$, $\theta^{(s)} = \Phi(\frac{1}{2}(\|u - u'\|/\beta)^{\alpha/2})$
- Estimate the spatial parameters by

$$\min_{\alpha,\beta} \sum_{s} \left\| \Phi\left(\frac{1}{2}(\|u - u'\|/\beta)^{\alpha/2}\right) - \widehat{\theta}^{(s)} \right\|^2$$

- In the paper, we prove joint asymptotic normality of the collection of estimators $\widehat{c}_n^{(s)}$
- Leads to CLT's for $\widehat{\theta}^{(s)}$ and for (α, β)

Some illustration

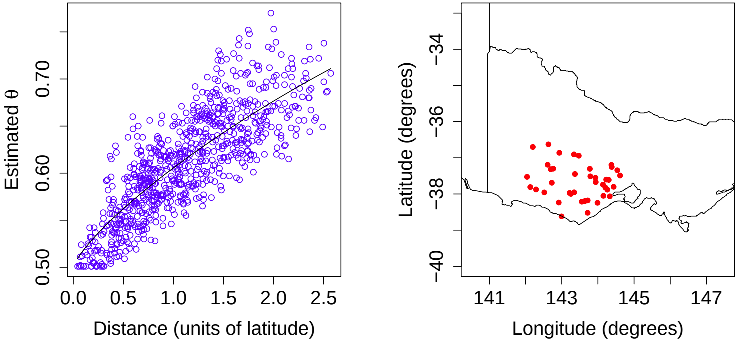

Figure: Left: Estimated parameters $\widehat{\theta}^{(s)}$ against the Euclidean distance. Right: The 40 sampled locations in the state of Victoria, southeastern Australia.

Thanks for your attention!

A few references

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